



**Dreidimensionale Finite-Element-Analysen zur  
Beurteilung der Sickerströmung und des Qualmwassers  
in ungesättigten und gesättigten oberflächennahen  
Erdschichten von Flussregionen**

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**IBSNM**

**Ingenieur-Büro Systemanalyse und Numerische Modellierung**

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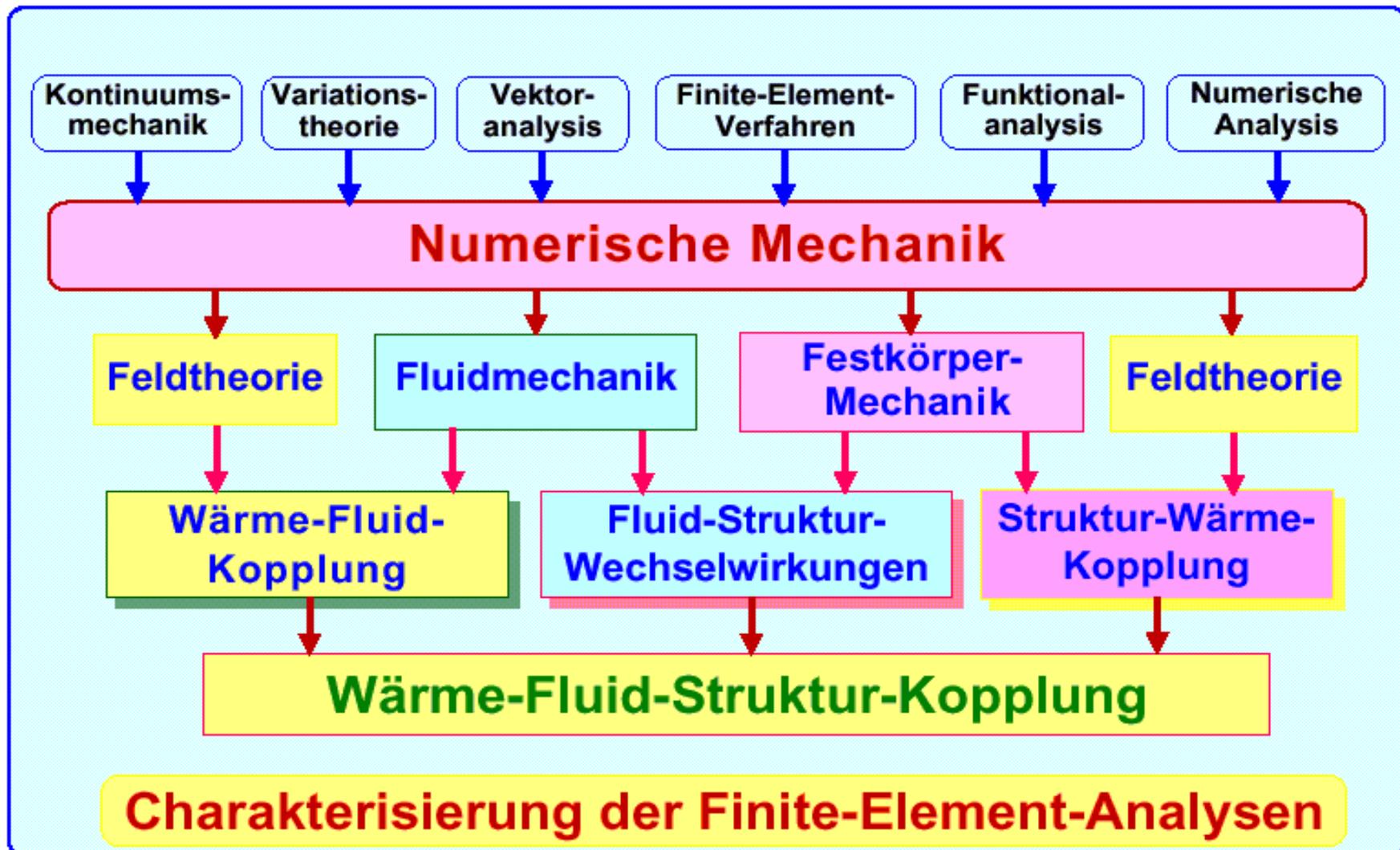
**Homepage: <http://www.rudolf-adolf-dietrich.de>**

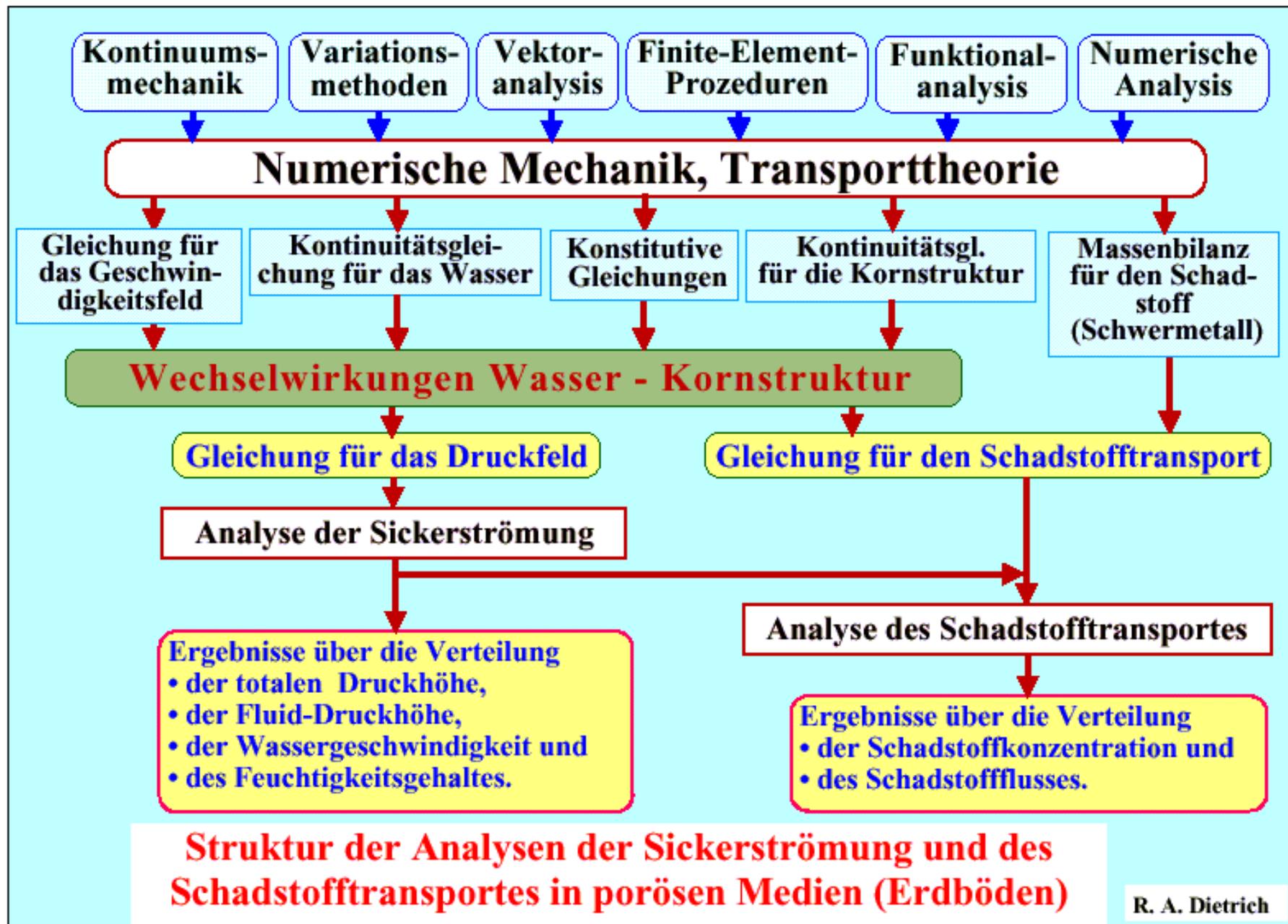
## Vorteile der FEM gegenüber der FDM

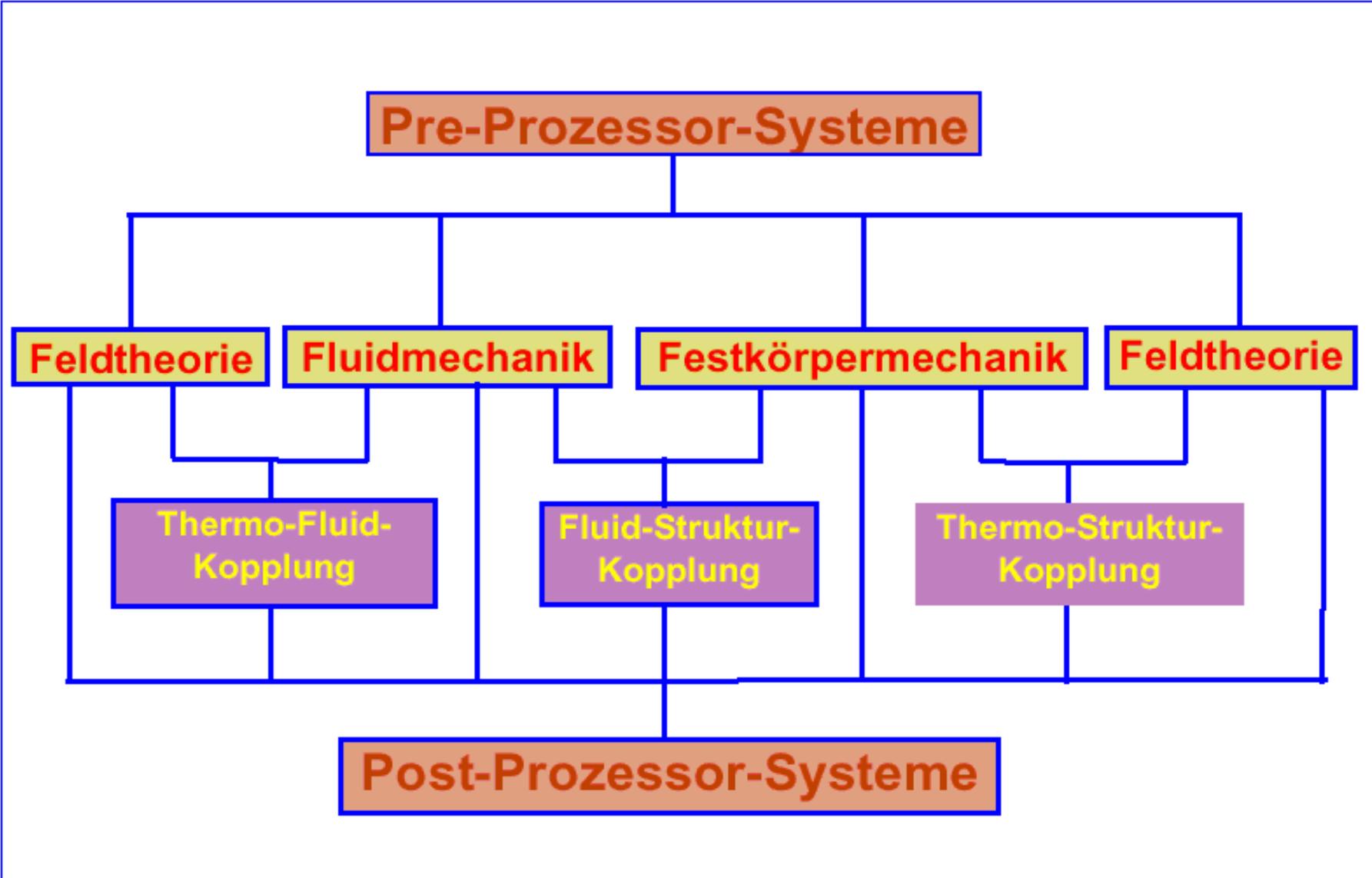
- **Bessere Erfassung von ungleichmäßigen Geometrien.**
- **Unterschiedliche Verfeinerung der Netze in verschiedenen Bereichen möglich.**
- **Heterogenität und Anisotropie sind leicht zu erfassen.**
- **Die Randbedingungen werden durch die integrale Formulierung direkt eingebunden.**

## **Fundamentale Unterschiede zwischen der FE- und FD-Methode**

- **Die FEM liefert räumlich kontinuierliche Lösungen.**
- **Die FDM gewährt Lösungen nur an diskreten Punkten.**







## Gleichung des Druckfeldes

Auf der Basis der Erhaltung der Masse gilt:

$$\mathbb{L}(h) := \Lambda \frac{\partial h}{\partial t} - \operatorname{div}[\mathbf{v}_D] = 0$$

$$\mathbb{L}(h) := \Lambda \frac{\partial h}{\partial t} - \operatorname{div}[\mathbf{K} \operatorname{grad} (h + z)] = 0$$

$$\Lambda = \frac{\Theta \rho_w g}{n_e} [\alpha + n_e \beta] + \frac{d\Theta}{dh}$$

## Galerkin-Finite-Element-Methode

**Gewichtetes Residuum**

$$\int_V w_i \mathcal{L}(h) dV = 0 \quad i = 1, 2, 3, 4$$

**Wichtungsfunktion**

$$w_i(y,z) = N_i(y,z) \quad i = 1, 2, \dots, N$$

## Finite-Element-Approximation

$${}^{t+\alpha\Delta t}\mathbf{S} \dot{\mathbf{h}}(t) + {}^{t+\alpha\Delta t}\mathbf{T} \mathbf{h}(t) + {}^{t+\alpha\Delta t}\mathbf{D} + {}^{t+\alpha\Delta t}\mathbf{Q} = \mathbf{0}$$

- Konzentrierte Speichermatrix :  $\forall i \neq j \rightarrow {}^{t+\alpha\Delta t}S_{ij} = 0$
- Konsistente Speichermatrix :  $\exists i \neq j \rightarrow {}^{t+\alpha\Delta t}S_{ij} > 0$

## Speicherungsmatrix

$$\mathbf{S}^{(m)} := \int_{V^{(m)}} \mathbf{N}^{(m)} \Lambda \mathbf{N}^{(m)T} dV^{(m)}$$

## Transmissionsmatrix

$$\mathbf{T}^{(m)} = \int_{V^{(m)}} \text{grad}(N^{(m)})^T \mathbf{K} \text{grad}(N^{(m)}) dV^{(m)}$$

$$T_{ij}^{(m)} = \int_{V^{(m)}} \begin{bmatrix} \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial z} \end{bmatrix} \begin{bmatrix} k_{yy} & k_{yz} \\ k_{zy} & k_{zz} \end{bmatrix} \begin{bmatrix} \frac{\partial N_j}{\partial y} \\ \frac{\partial N_j}{\partial z} \end{bmatrix} dV^{(m)}$$

## Transmissionsmatrix

$${}^e T_{ij} = \int_{Re} \left[ \frac{\partial N}{\partial y} \cdot \left( k_{yy} \frac{\partial N_j}{\partial y} + k_{yz} \frac{\partial N_j}{\partial z} \right) + \frac{\partial N_i}{\partial z} \cdot \left( k_{zy} \frac{\partial N_j}{\partial y} + k_{zz} \frac{\partial N_j}{\partial z} \right) \right] dRe$$

## Geodätischer Lastvektor

$$\mathbf{D}^{(m)} = \int_{V^{(m)}} \text{grad}(N^{(m)})^T \mathbf{K} \text{grad}(z) dV^{(m)}$$

$$D_i^{(m)} = \int_{V^{(m)}} \left[ \frac{\partial N_i}{\partial y} \quad \frac{\partial N_i}{\partial z} \right] \begin{bmatrix} k_{yy} & k_{yz} \\ k_{zy} & k_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dV^{(m)}$$

## Lastvektor für den Wasserstrom am Rand

$$\mathbf{Q}^{(m)} = \int_{\Gamma^{(m)}} \mathbf{n} \cdot \mathbf{N}^{(m)\top} \mathbf{K} \text{grad}(h + z) \, d\Gamma^{(m)}$$

$$Q_i^{(m)} = \int_{\Gamma^{(m)}} N_i \begin{bmatrix} k_{yz} & k_{zz} \end{bmatrix} \begin{bmatrix} \frac{\partial(h+z)}{\partial y} \\ \frac{\partial(h+z)}{\partial z} \end{bmatrix} d\Gamma^{(m)}$$

# Grundgleichung des Schadstofftransportes

## Prinzip zur Erhaltung der Masse

$$L(c_g) := \underbrace{\frac{\partial(\Theta c_g + \rho_T s_a)}{\partial t}}_{\text{zeitliche Änderung des Schadstoffgehaltes}} + \underbrace{\text{div}[(\Theta c_g + \rho_T s_a) \mathbf{v}_s]}_{\text{zeitliche Ausbreitung des Schadstoffgehaltes}} - \underbrace{\text{div}[\Theta \mathbf{D} \cdot \text{grad } c_g]}_{\text{hydraulische Dispersion und molekularer Diffusionsfluß}}$$

$$+ \underbrace{\text{div}[c_g \mathbf{v}_D]}_{\text{Advektion}} + \underbrace{\lambda(\Theta c_g + \rho_T s_a)}_{\text{Zerfall}} = 0$$

## Galerkin-Finite-Element-Methode

**Gewichtetes Residuum**

$$\int_V w_i \mathcal{L}(c) dV = 0 \quad i = 1, 2, 3, 4$$

**Wichtungsfunktion**

$$w_i(y,z) = N_i(y,z) \quad i = 1, 2, \dots, N$$

## Finite-Element-Approximation

$${}^{t+\alpha\Delta t}\mathbf{E} \dot{\mathbf{C}}(t) + {}^{t+\alpha\Delta t}\mathbf{F} \mathbf{C}(t) + {}^{t+\alpha\Delta t}\mathbf{G} = \mathbf{0}$$

- Konzentrierte Speichermatrix:  $\forall i \neq j \rightarrow {}^{t+\Delta t}E_{ij} = 0$
- Konsistente Speichermatrix:  $\exists i \neq j \rightarrow {}^{t+\Delta t}E_{ij} > 0$

## Speicherungsmatrix

$$\mathbf{E}^{(m)} = \int_{V^{(m)}} \mathbf{N}^{(m)} \mathbf{A} \mathbf{N}^{(m)T} dV^{(m)}$$

## Transmissionsmatrix

$$\begin{aligned} \mathbf{F}^{(m)} = & \int_{V^{(m)}} \text{grad}(\mathbf{N}^{(m)})^T \Theta \mathbf{D} \text{grad}(\mathbf{N}^{(m)}) dV^{(m)} \\ & - \int_{V^{(m)}} \text{grad}(\mathbf{N}^{(m)})^T \mathbf{v}_D \mathbf{N}^{(m)T} dV^{(m)} \\ & + \int_{V^{(m)}} \mathbf{N}^{(m)} \mathbf{B} \mathbf{N}^{(m)T} dV^{(m)} \end{aligned}$$

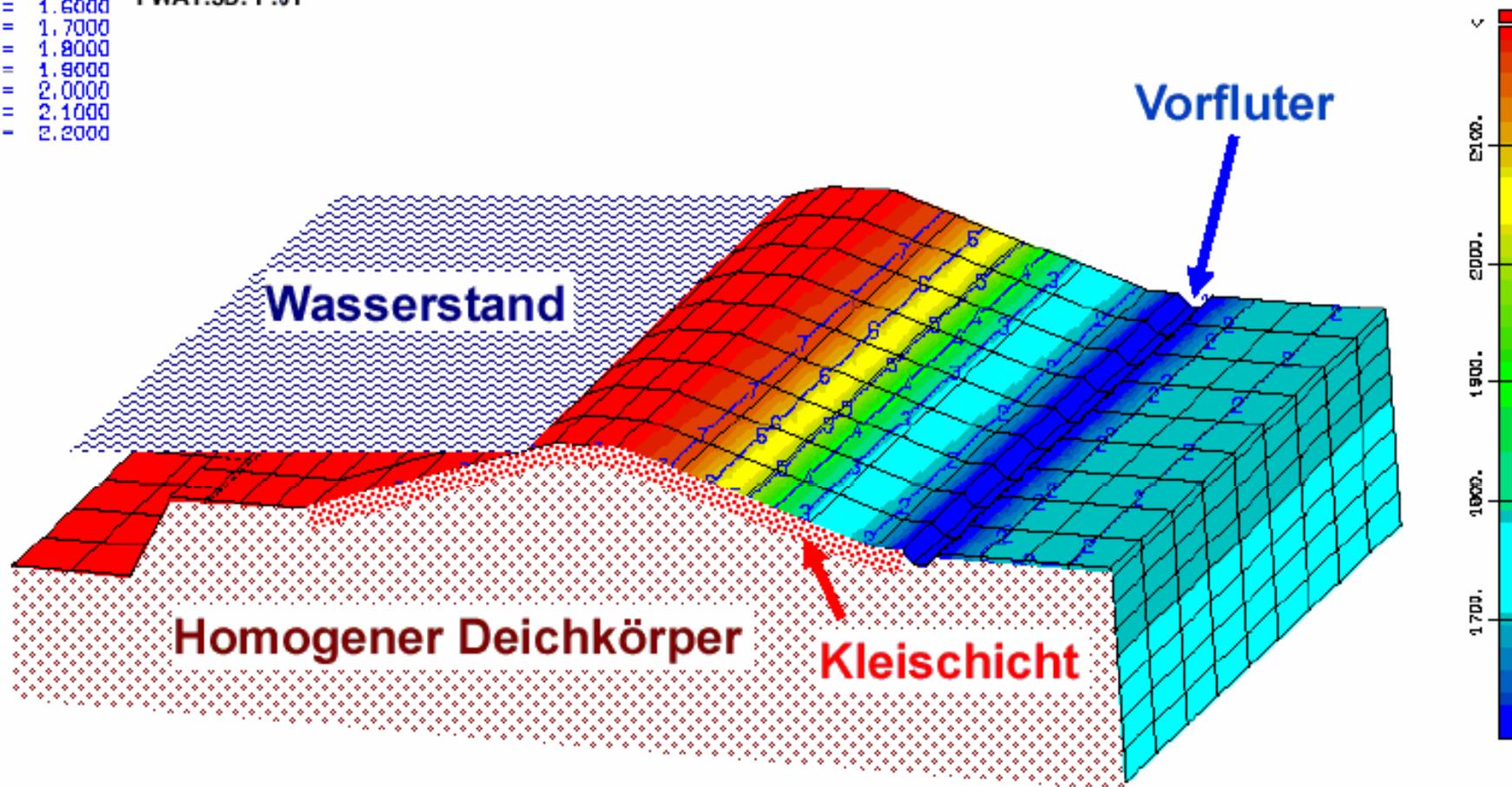
## Lastvektor für die Flüsse am Rand

$$\mathbf{G}^{(m)} = \int_{\Gamma^{(m)}} \mathbf{n} \cdot \mathbf{N}^{(m)} (c_g \mathbf{v}_D) d\Gamma^{(m)}$$
$$- \int_{\Gamma^{(m)}} \mathbf{n} \cdot \mathbf{N}^{(m)} \Theta \mathbf{v}_D \text{grad}(c_g) d\Gamma^{(m)}$$

Faktor:  $10^3$   
1 = 1.6000  
2 = 1.7000  
3 = 1.8000  
4 = 1.9000  
5 = 2.0000  
6 = 2.1000  
7 = 2.2000

FWAT.3D: P.01

IBSNM/20.12.02



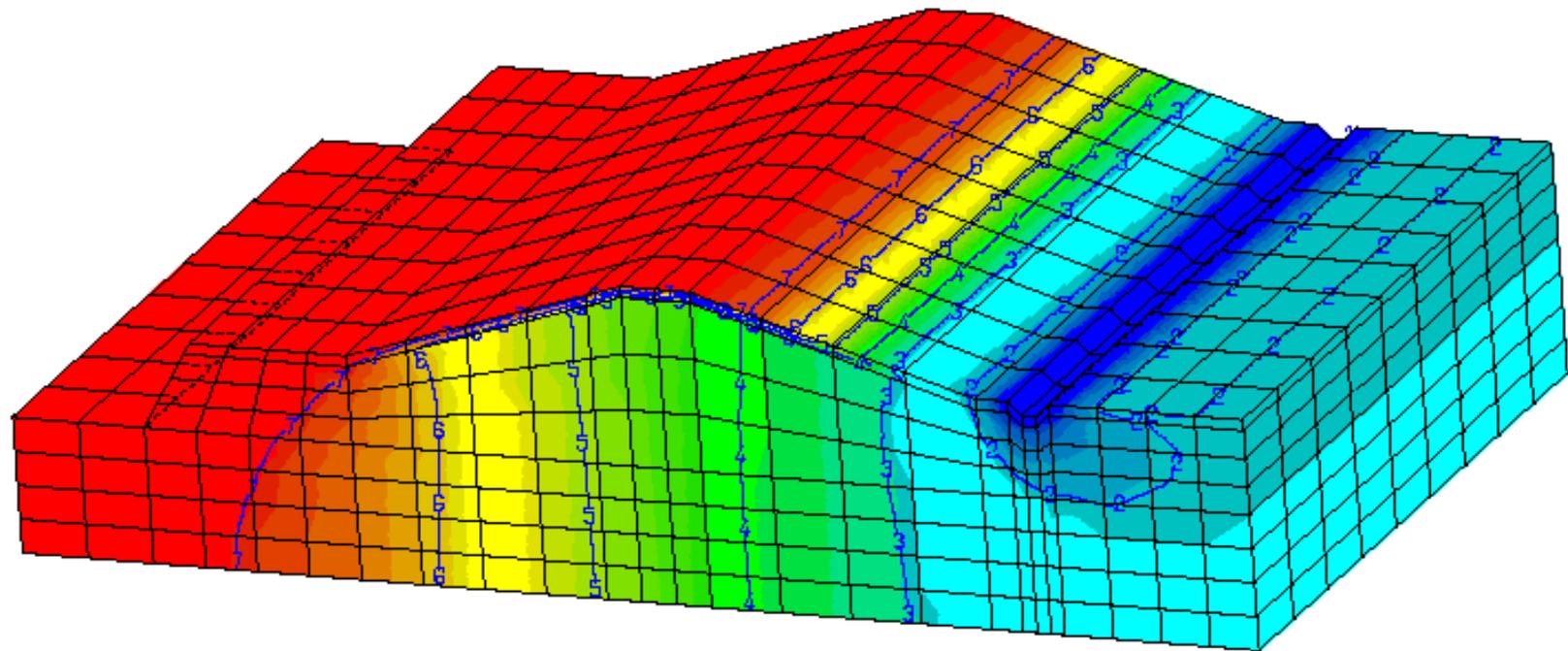
# Systemskizze

Z  
Y  
X  
R.A. Dietrich

Faktor:  $10^3$   
1 = 1.6000  
2 = 1.7000  
3 = 1.8000  
4 = 1.9000  
5 = 2.0000  
6 = 2.1000  
7 = 2.2000

FWAT.3D: P.01

IBSNM/20.12.02



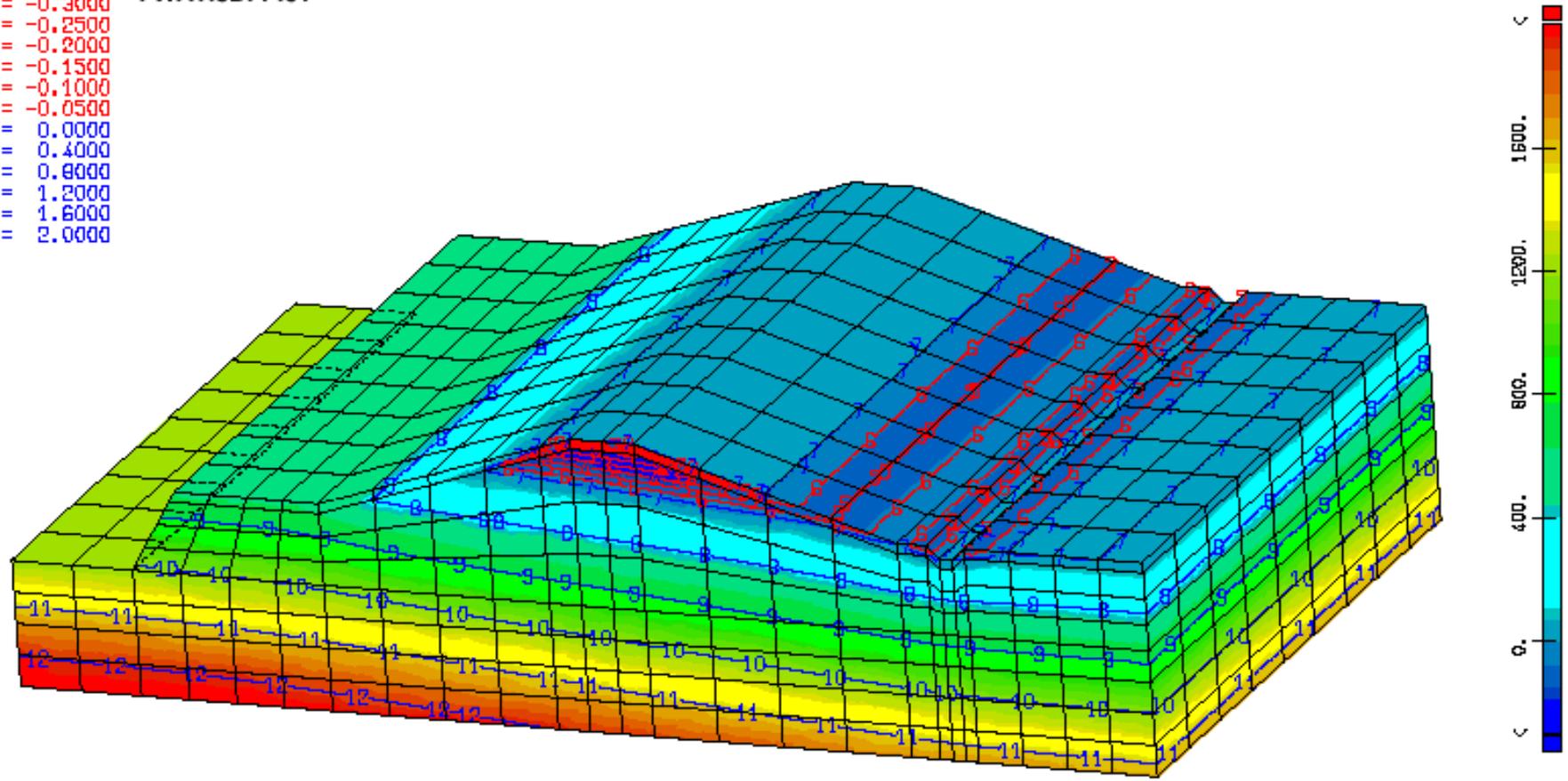
**BILD : TOTALE DRUCKHÖHE (cm)**  
Stationäre Analyse

Z  
Y  
X  
R.A. Dietrich

Faktor:  $10^3$   
 1 = -0.3000  
 2 = -0.2500  
 3 = -0.2000  
 4 = -0.1500  
 5 = -0.1000  
 6 = -0.0500  
 7 = 0.0000  
 8 = 0.4000  
 9 = 0.8000  
 10 = 1.2000  
 11 = 1.6000  
 12 = 2.0000

FWAT.3D: P.01

IBSNM/20.12.02



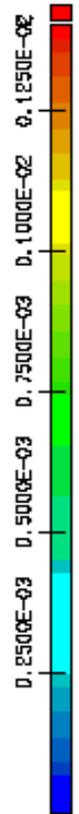
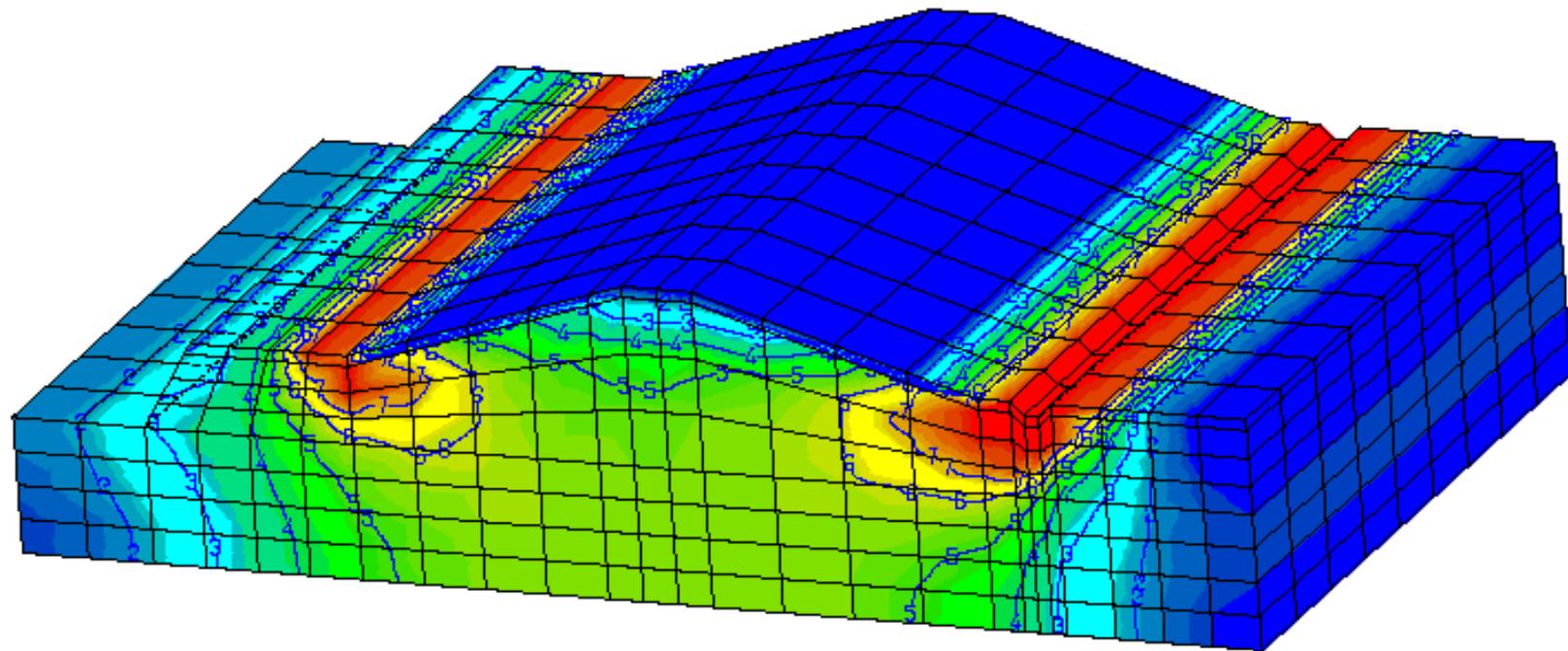
**BILD : FLUID-DRUCKHÖHE (cm)**  
 Stationäre Analyse

Z  
 Y  
 X  
 R.A. Dietrich

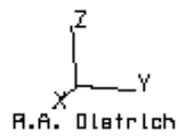
Faktor:  $10^{-3}$   
1 = 0.0000  
2 = 0.2000  
3 = 0.4000  
4 = 0.6000  
5 = 0.8000  
6 = 1.0000  
7 = 1.2000

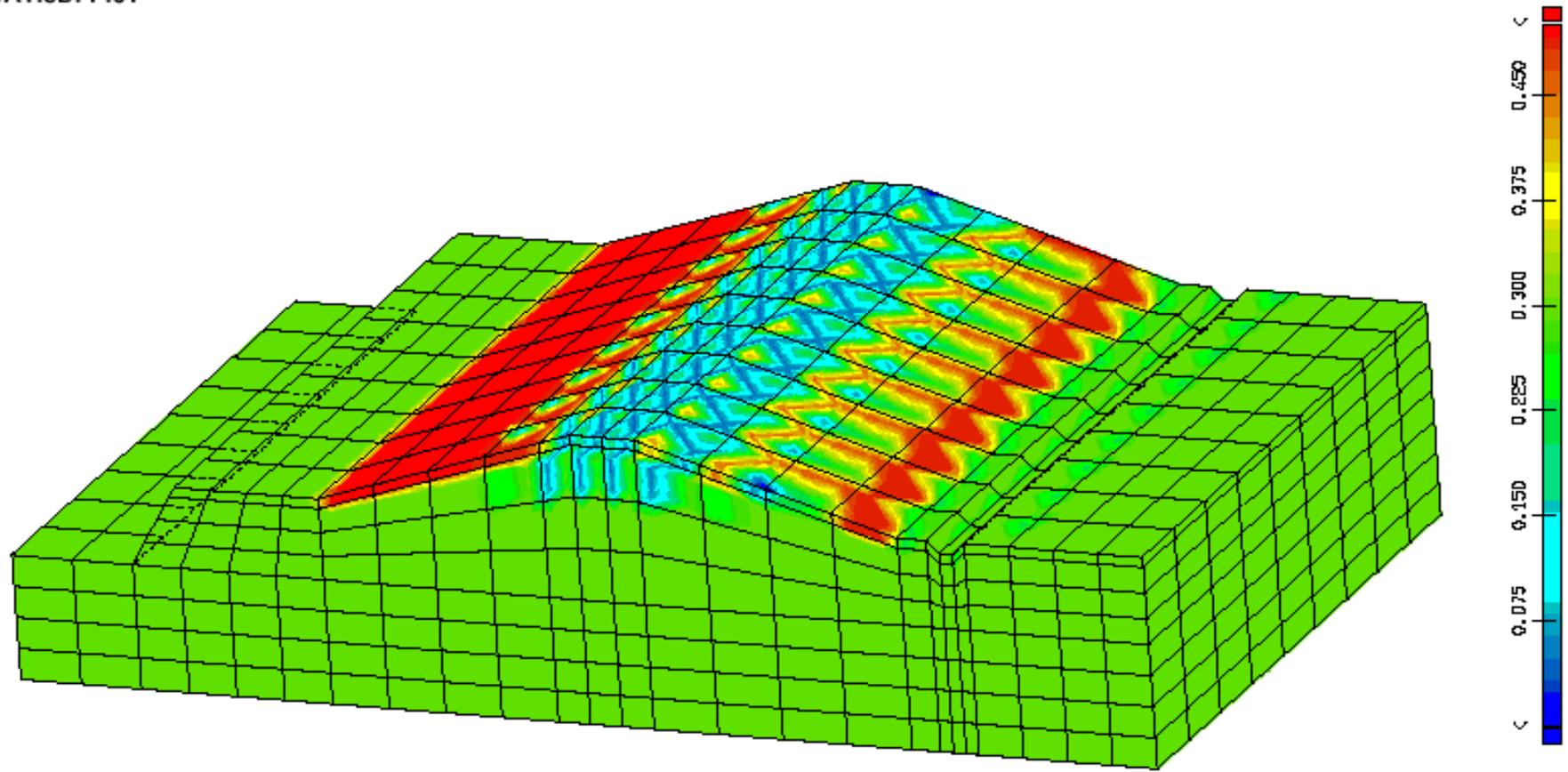
FWAT.3D: P.01

IBSNM/20.12.02



**BILD : WASSERGESCHWINDIGKEIT (cm/s)**  
Stationäre Analyse





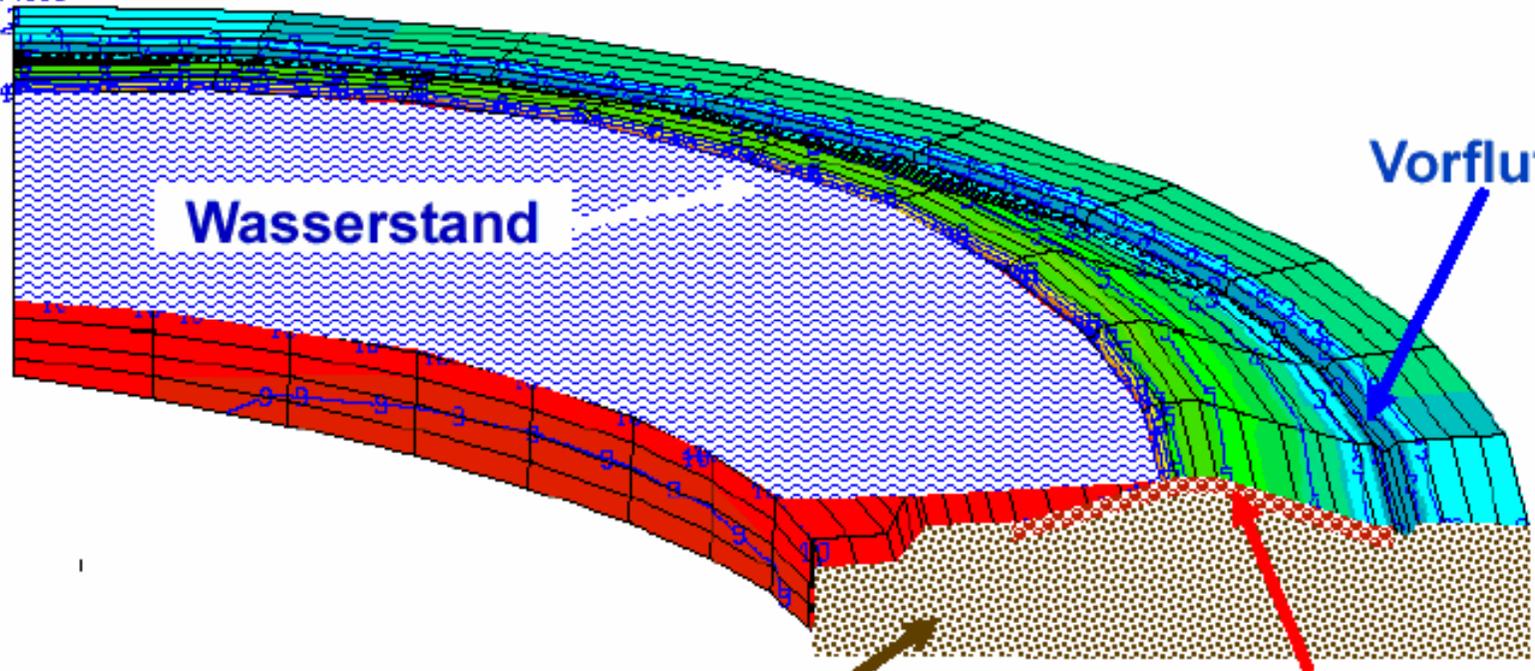
**BILD : FEUCHTIGKEITSGEHALT (-)**  
Stationäre Analyse

Z  
Y  
X  
R.A. Dietrich

- Faktor:  $10^2$
- 1 = 0.0000
  - 2 = 1.0000
  - 3 = 2.0000
  - 4 = 3.0000
  - 5 = 4.0000
  - 6 = 5.0000
  - 7 = 6.0000
  - 8 = 7.0000
  - 9 = 7.2000
  - 10 = 7.4000

DEICH-3D:P.01

IBSNM / 14.07.03



Homogener Deichkörper

Kleischicht

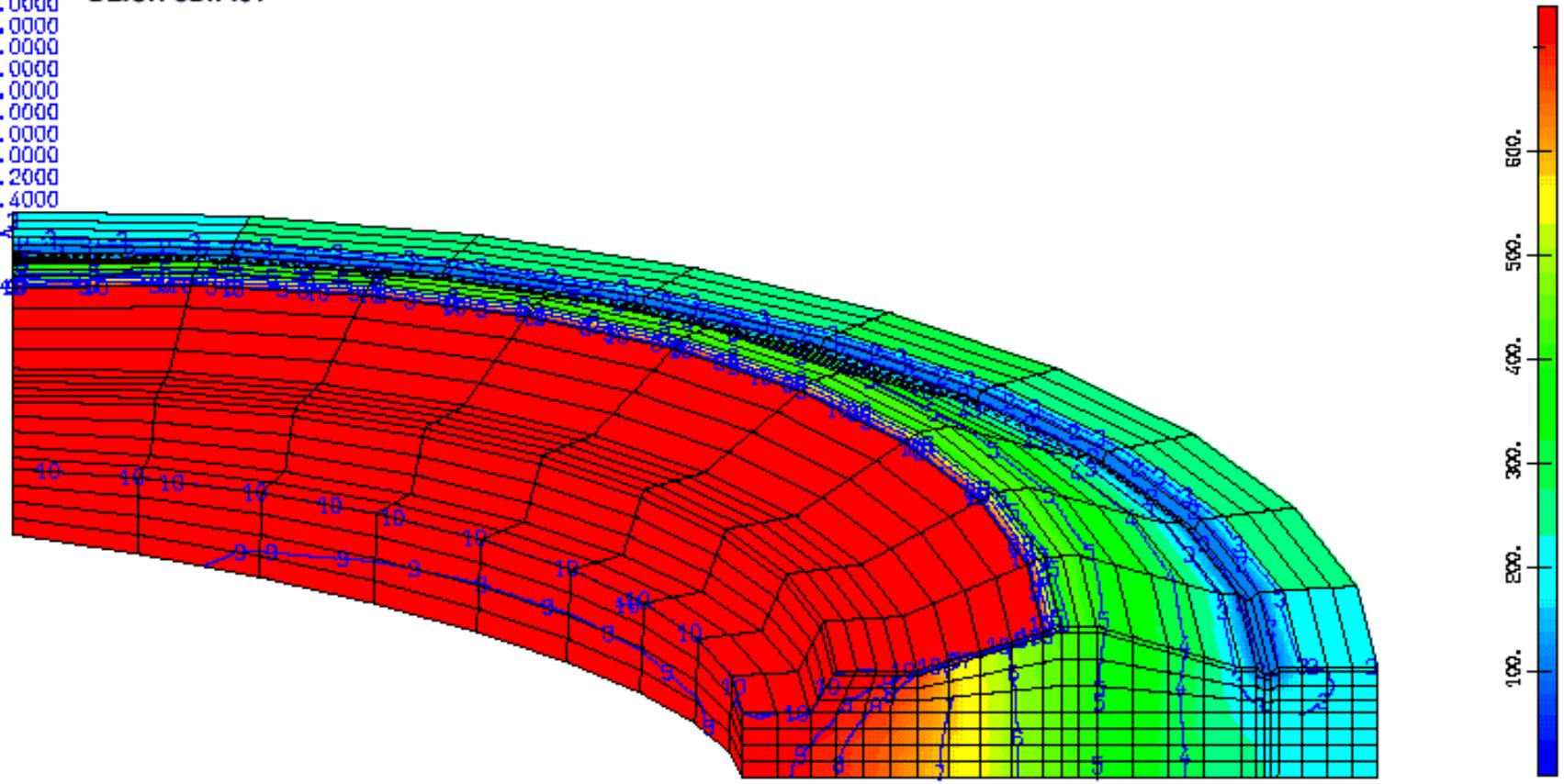
# Systemskizze

Z  
Y  
R.A. Dietrich

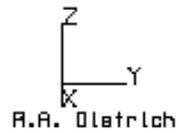
- Faktor:  $10^2$
- 1 = 0.0000
  - 2 = 1.0000
  - 3 = 2.0000
  - 4 = 3.0000
  - 5 = 4.0000
  - 6 = 5.0000
  - 7 = 6.0000
  - 8 = 7.0000
  - 9 = 7.2000
  - 10 = 7.4000

DEICH-3D:P.01

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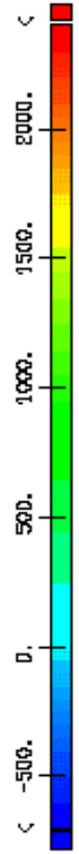
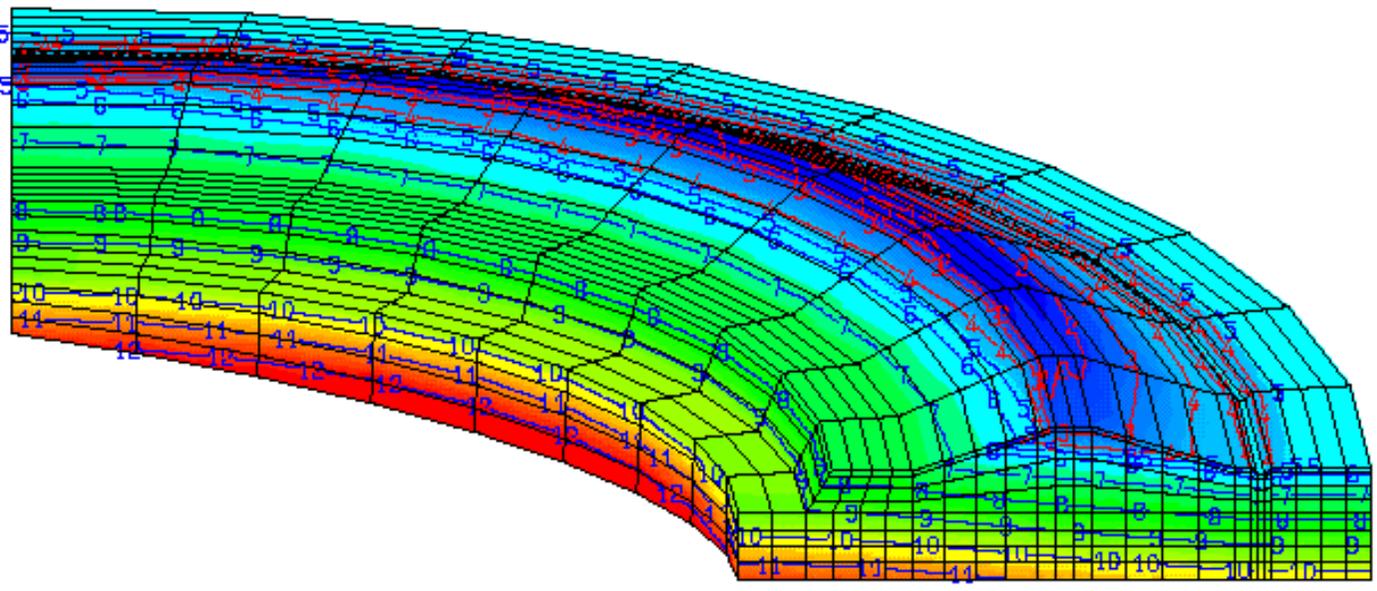
**TOTALE DRUCKHÖHE (cm)**  
Stationäre Analyse



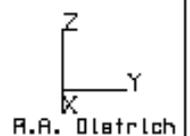
- Faktor:  $10^3$
- 1 = -0.7000
  - 2 = -0.5000
  - 3 = -0.3000
  - 4 = -0.1000
  - 5 = 0.0000
  - 6 = 0.1000
  - 7 = 0.4000
  - 8 = 0.8000
  - 9 = 1.2000
  - 10 = 1.6000
  - 11 = 2.0000
  - 12 = 2.4000

DEICH-3D:P.01

IBSNM / 14.07.03



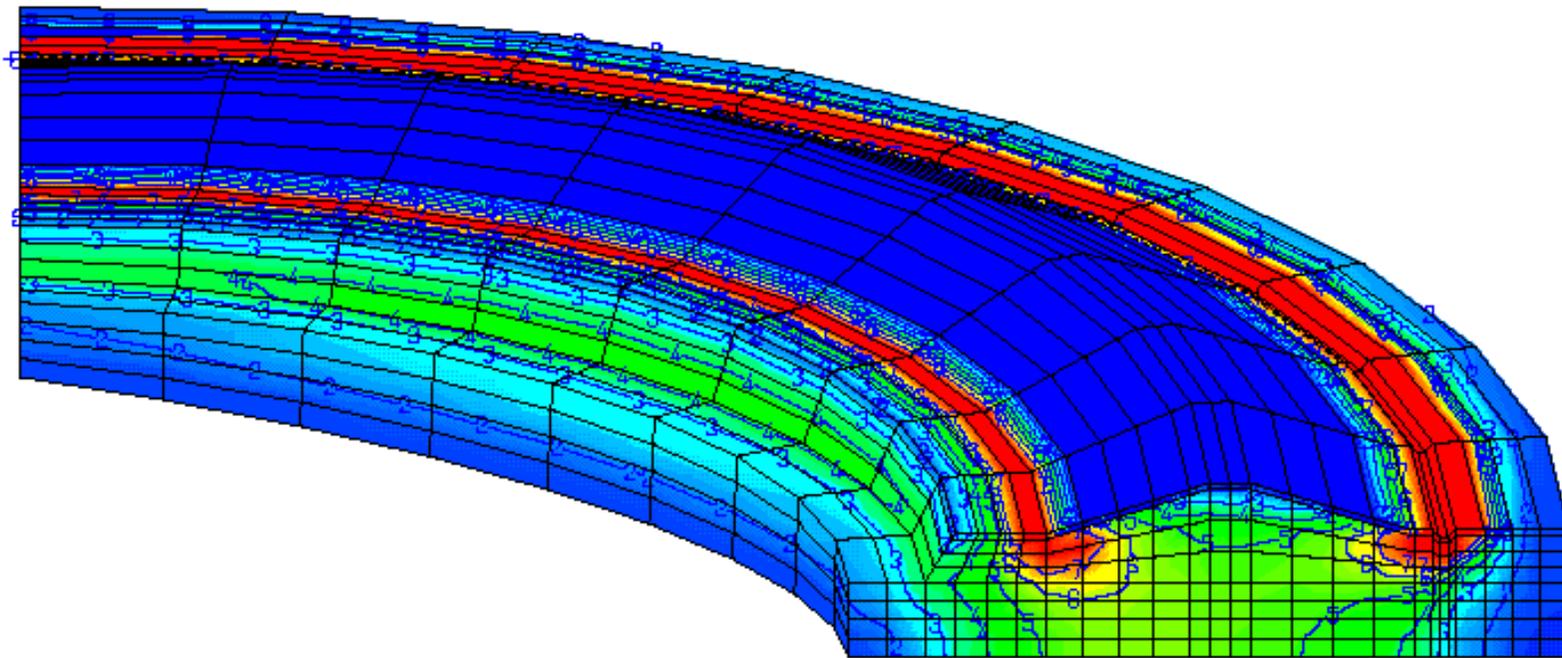
**FLUID-DRUCKHÖHE (cm)**  
Stationäre Analyse



Faktor:  $10^{-3}$   
1 = 0.0000  
2 = 0.2000  
3 = 0.4000  
4 = 0.6000  
5 = 0.8000  
6 = 1.0000  
7 = 1.2000

DEICH-3D:P.01

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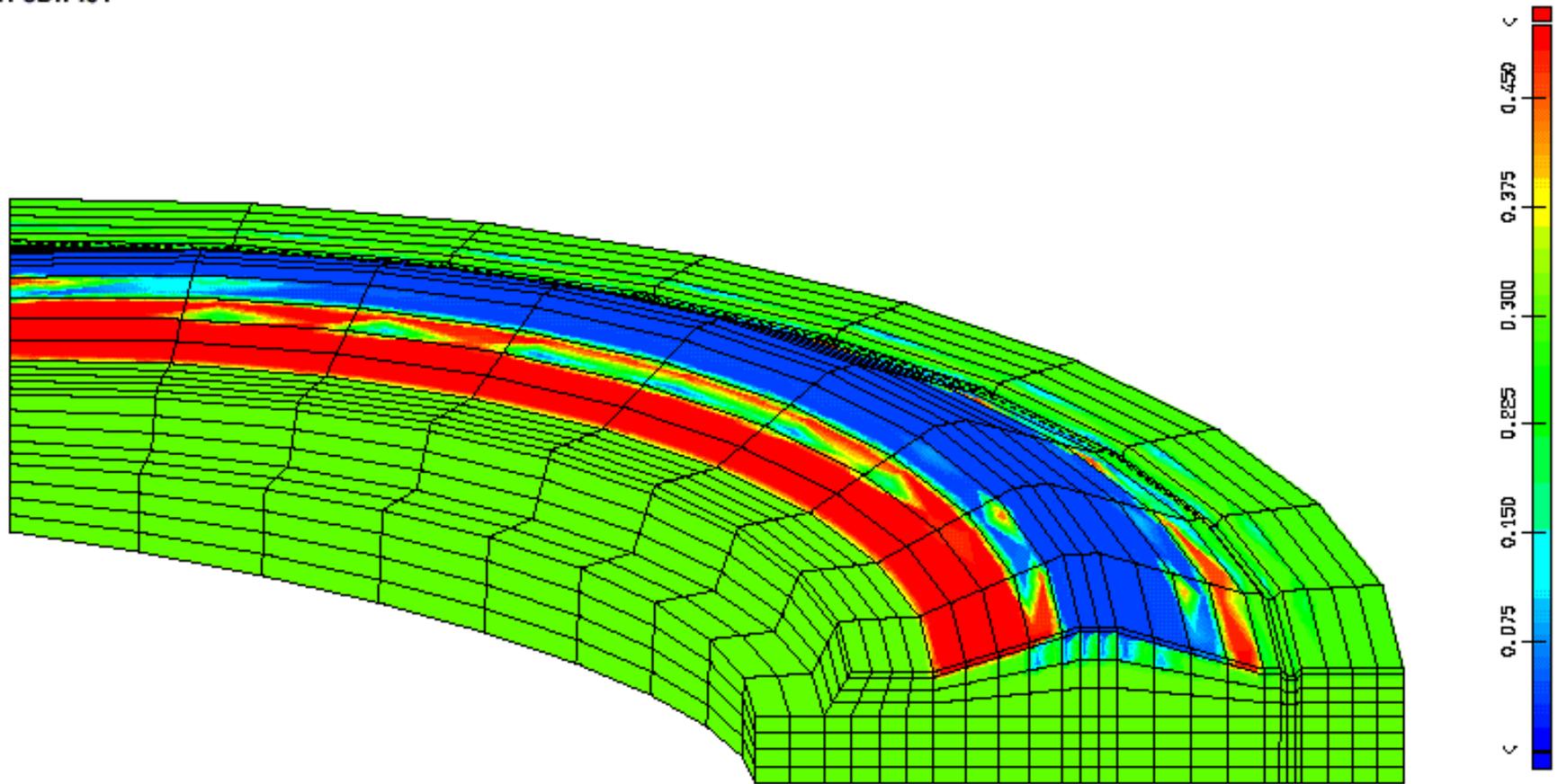


**WASSERGE SCHWINDIGKEIT (cm/s)**  
Stationäre Analyse

Z  
Y  
K  
R.A. Dietrich

DEICH-3D:P.01

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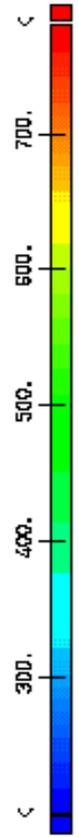
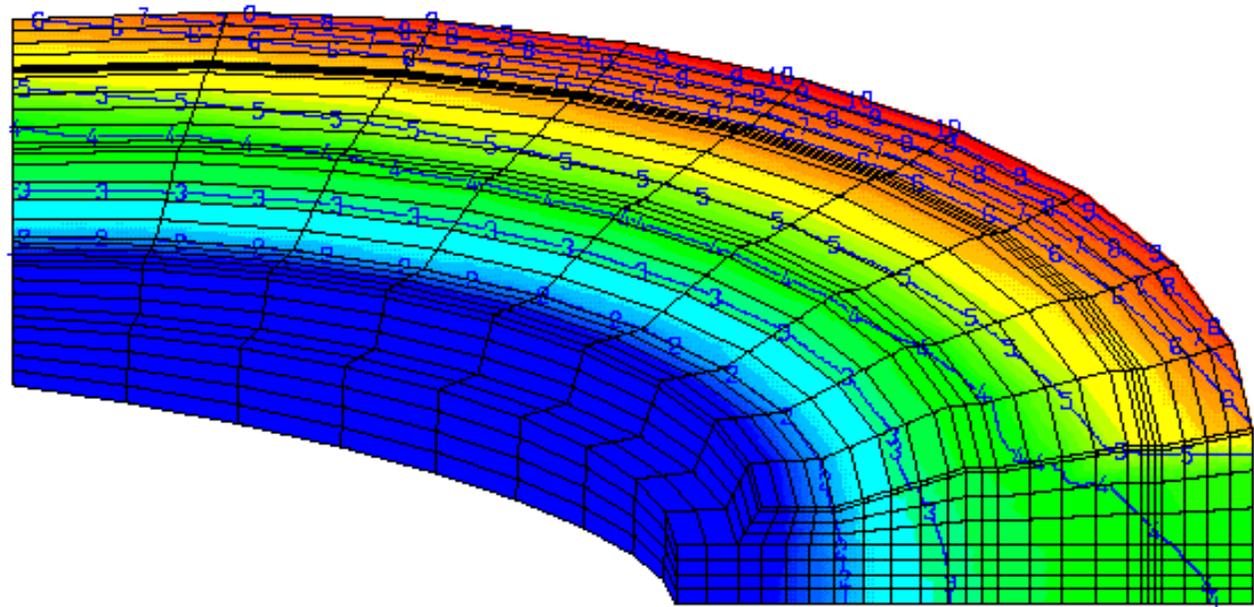
**FEUCHTIGKEITSGEHALT (-)**  
Stationäre Analyse

Z  
Y  
K  
R.A. Dietrich

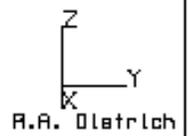
- Faktor:  $10^2$
- 1 = 2.0000
  - 2 = 3.0000
  - 3 = 4.0000
  - 4 = 5.0000
  - 5 = 6.0000
  - 6 = 7.0000
  - 7 = 7.2000
  - 8 = 7.4000
  - 9 = 7.6000
  - 10 = 7.8000

HANG-3D:P.01

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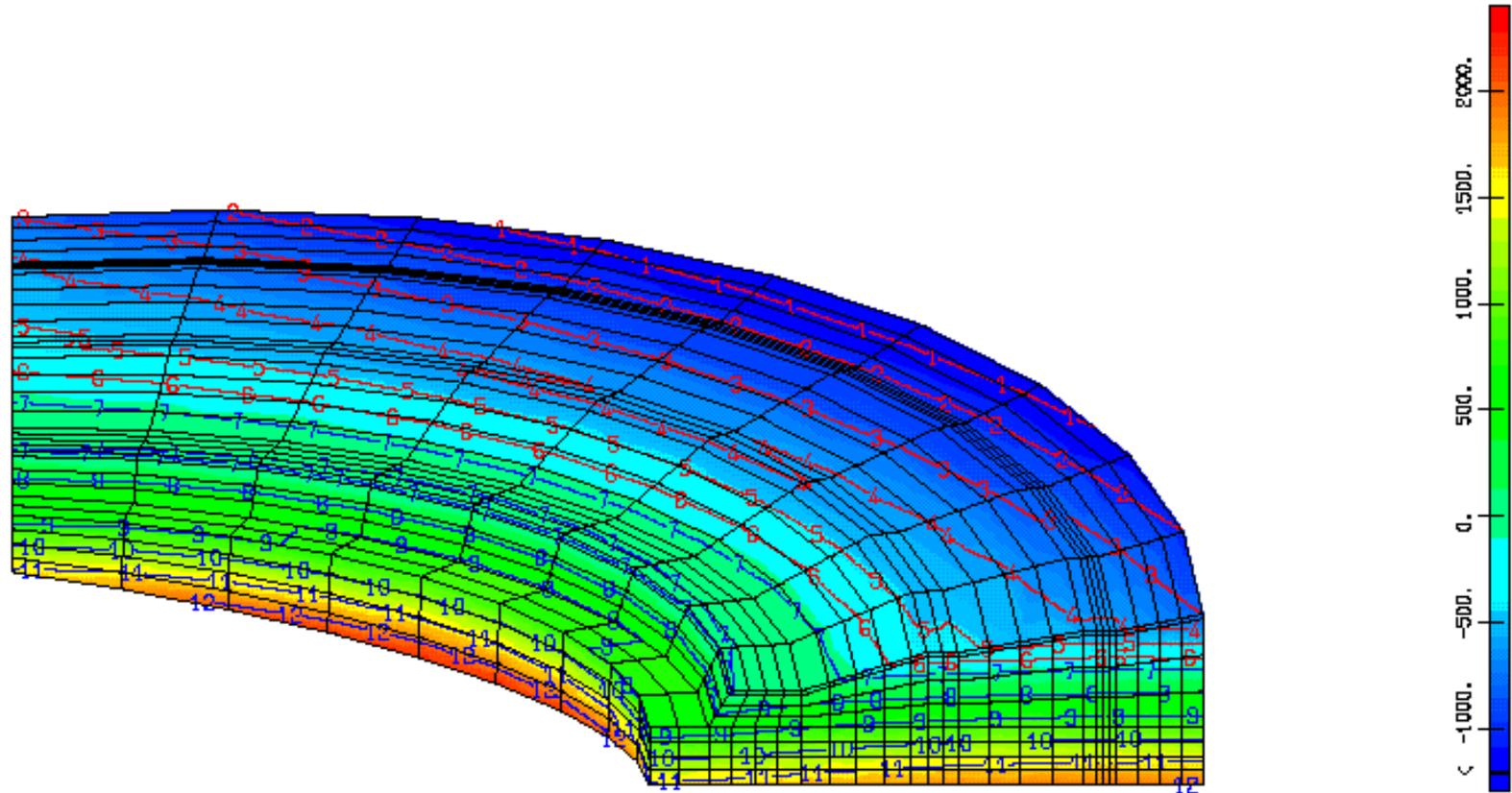
**TOTALE DRUCKHÖHE (cm)**  
Stationäre Analyse



Faktor:  $10^3$   
1 = -1.2000  
2 = -1.0000  
3 = -0.8000  
4 = -0.6000  
5 = -0.4000  
6 = -0.2000  
7 = 0.0000  
8 = 0.4000  
9 = 0.8000  
10 = 1.2000  
11 = 1.6000  
12 = 2.0000  
13 = 2.4000

HANG-3D:P.01

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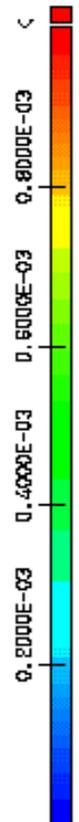
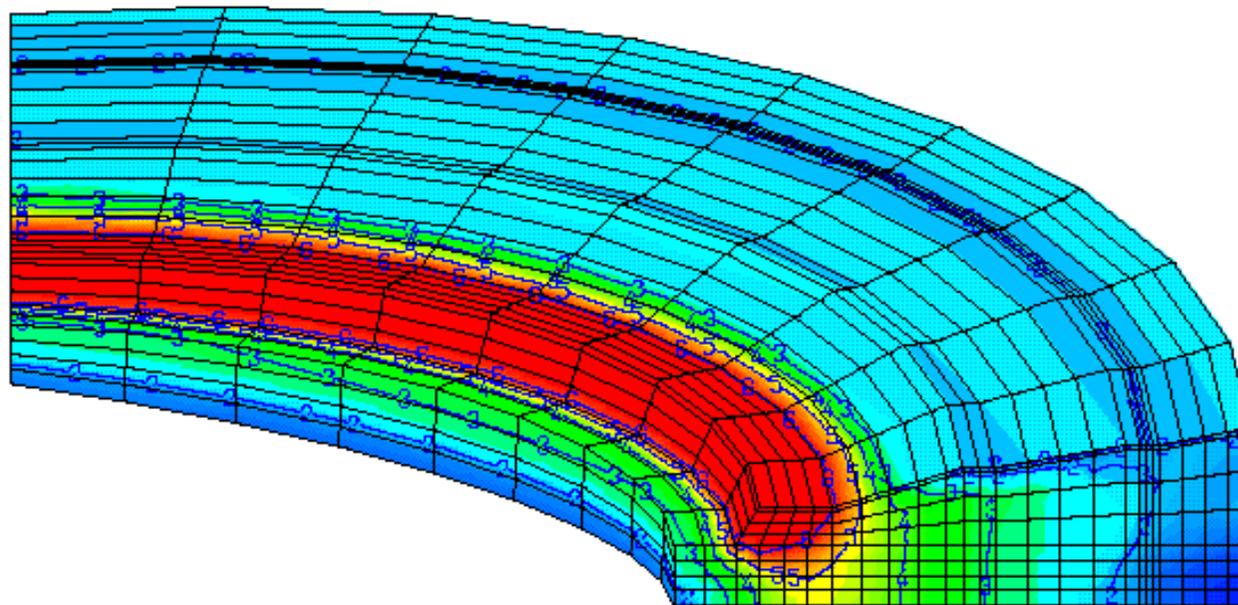
**FLUID-DRUCKHÖHE (cm)**  
Stationäre Analyse

Z  
Y  
X  
R.A. Dietrich

Faktor:  $10^{-4}$   
1 = 0.0000  
2 = 2.0000  
3 = 4.0000  
4 = 6.0000  
5 = 8.0000  
6 = 10.0000

HANG-3D:P.01

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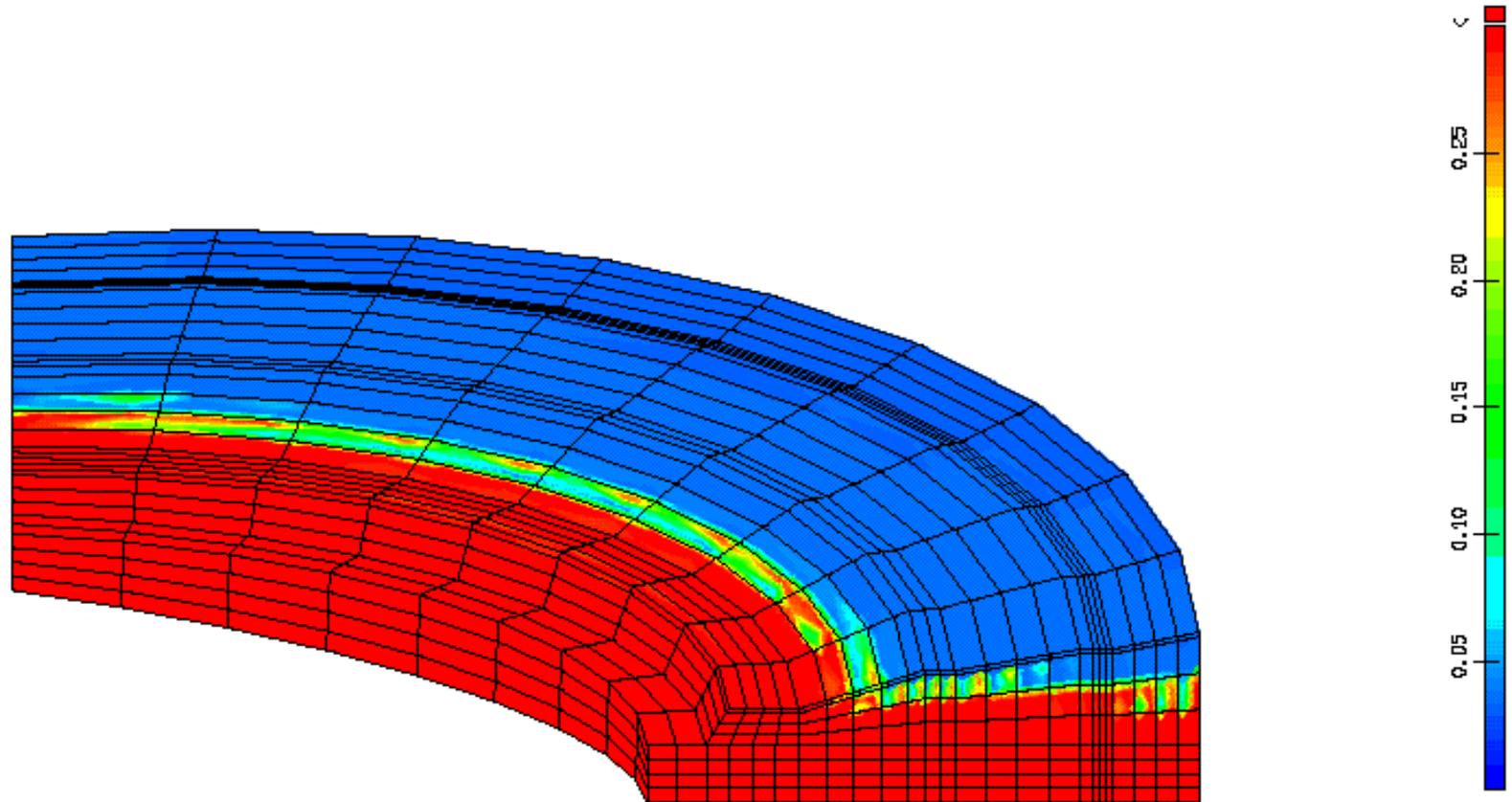


**WASSERGESCHWINDIGKEIT (cm/s)**  
Stationäre Analyse

Z  
Y  
X  
R.A. Dietrich

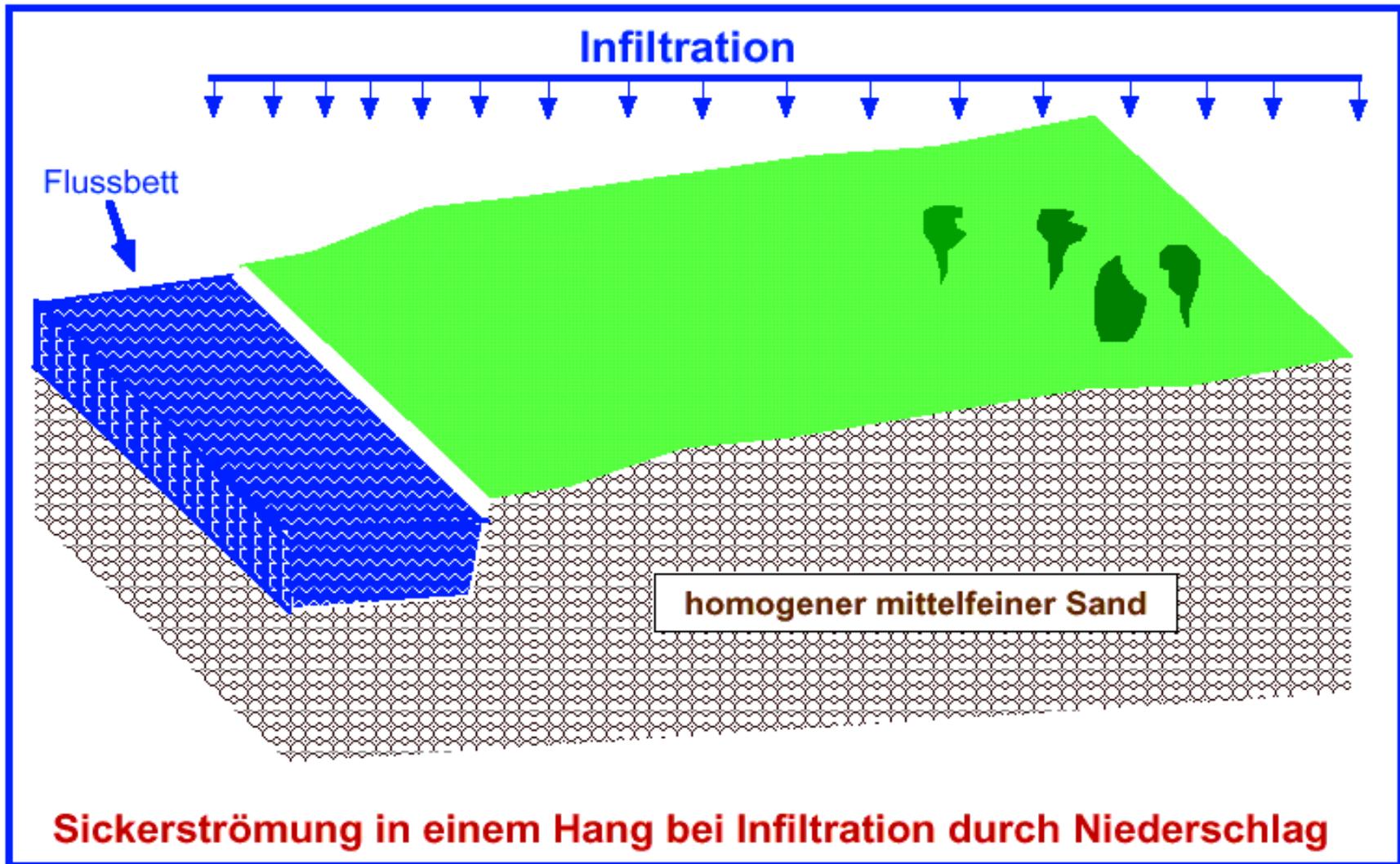
HANG-3D:P.01

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**FEUCHTIGKEITSGEHALT (-)**  
Stationäre Analyse

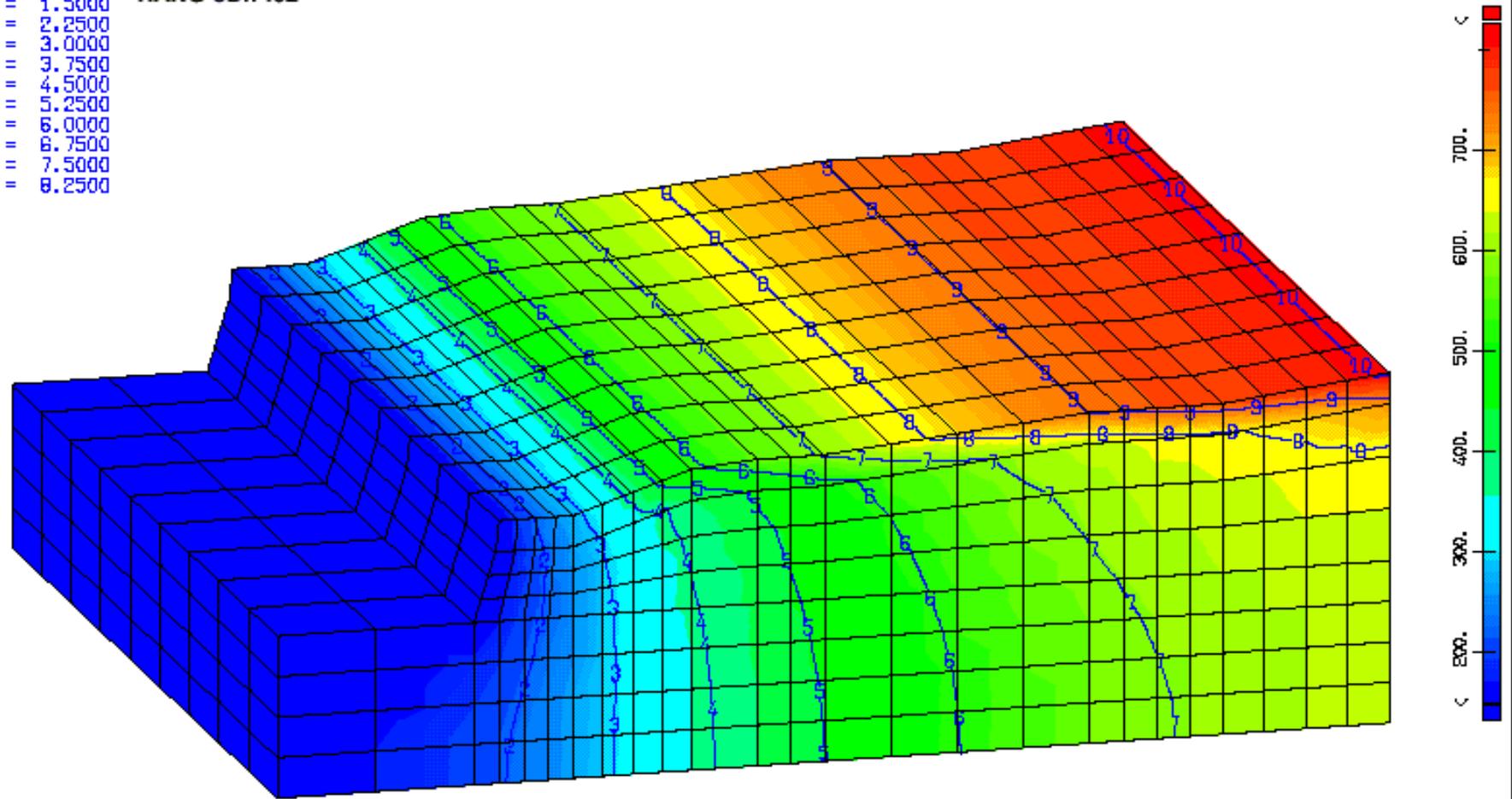
Z  
Y  
X  
R.A. Dietrich



- Faktor:  $10^2$
- 1 = 1.5000
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  - 3 = 3.0000
  - 4 = 3.7500
  - 5 = 4.5000
  - 6 = 5.2500
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  - 8 = 6.7500
  - 9 = 7.5000
  - 10 = 8.2500

HANG-3D:P.02

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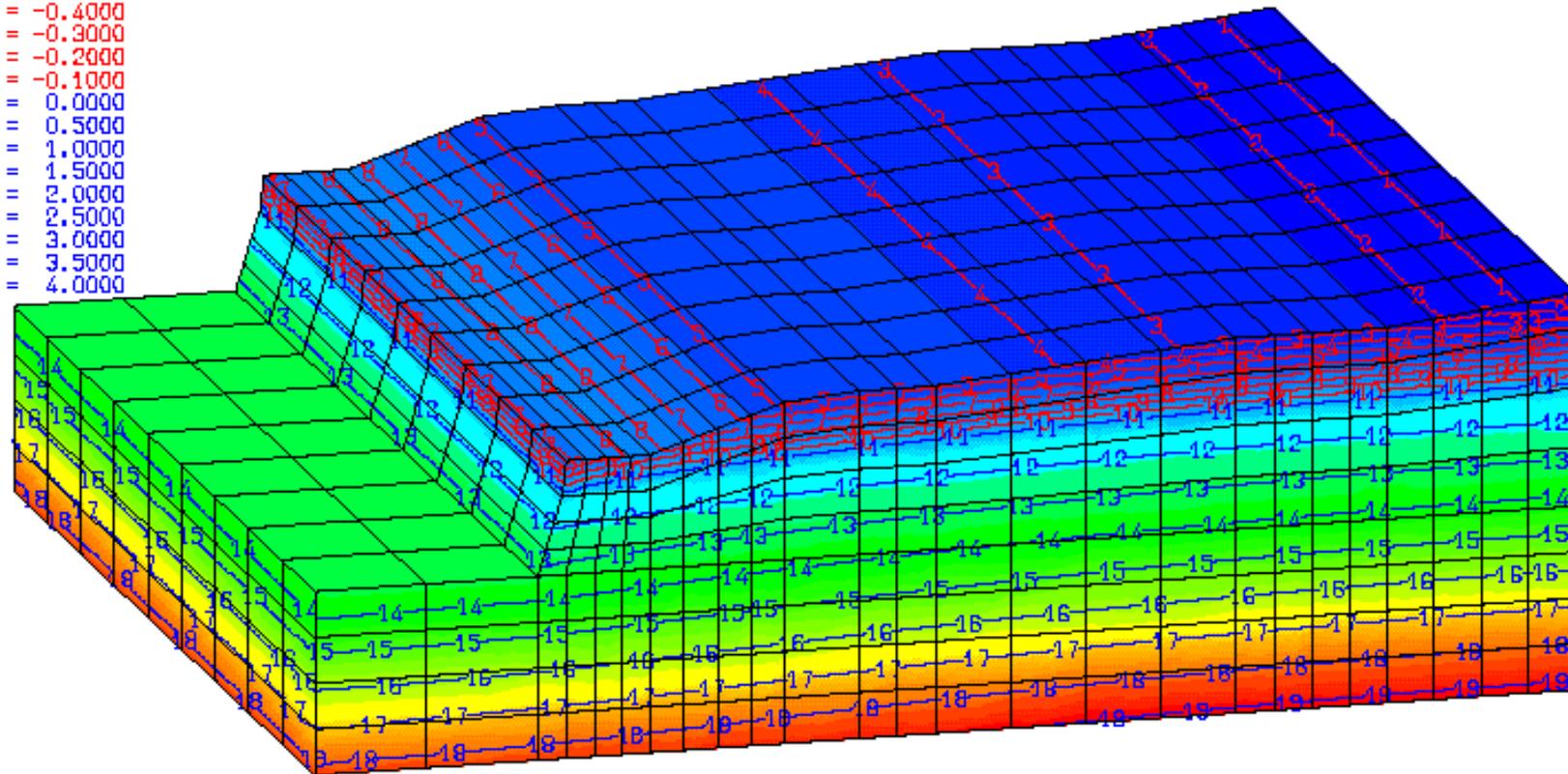
**TOTALE DRUCKHÖHE (cm)**  
Stationäre Analyse

Z  
Y  
R.A. Dietrich

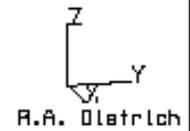
- Faktor:  $10^3$
- 1 = -1.0000
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  - 3 = -0.8000
  - 4 = -0.7000
  - 5 = -0.6000
  - 6 = -0.5000
  - 7 = -0.4000
  - 8 = -0.3000
  - 9 = -0.2000
  - 10 = -0.1000
  - 11 = 0.0000
  - 12 = 0.5000
  - 13 = 1.0000
  - 14 = 1.5000
  - 15 = 2.0000
  - 16 = 2.5000
  - 17 = 3.0000
  - 18 = 3.5000
  - 19 = 4.0000

HANG-3D:P.02

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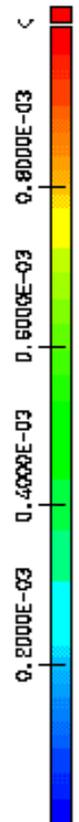
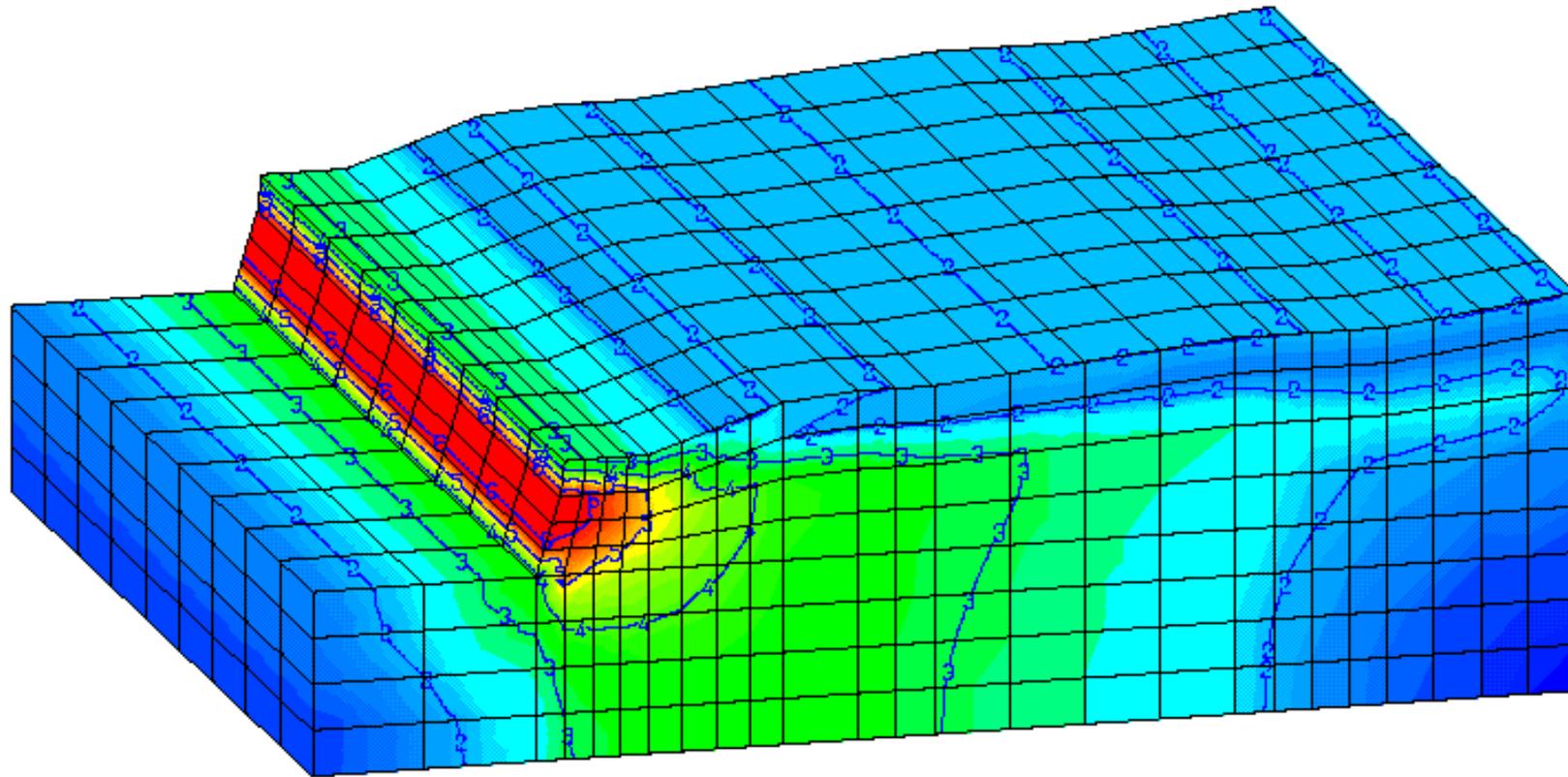
**FLUID-DRUCKHÖHE (cm)**  
Stationäre Analyse



Faktor:  $10^{-4}$   
1 = 0.0000  
2 = 2.0000  
3 = 4.0000  
4 = 6.0000  
5 = 8.0000  
6 = 10.0000

HANG-3D:P.02

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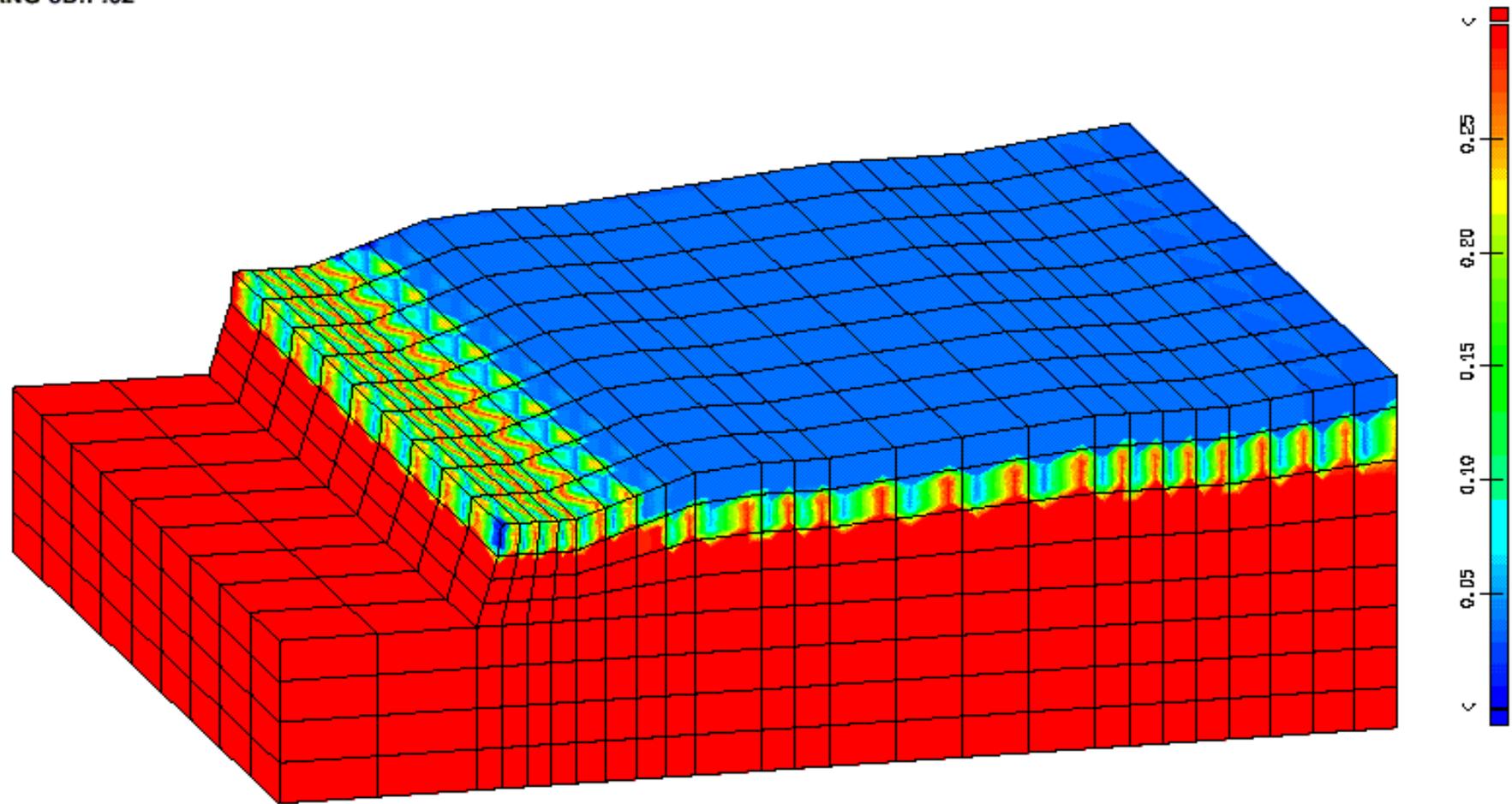


**WASSERGESCHWINDIGKEIT (cm/s)**  
Stationäre Analyse

Z  
Y  
X  
R.A. Dietrich

HANG-3D:P.02

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**FEUCHTIGKEITSGEHALT (-)**  
Stationäre Analyse

Z  
Y  
X  
R.A. Dietrich



**Vielen Dank für Ihr Interesse  
und für Ihre Aufmerksamkeit**